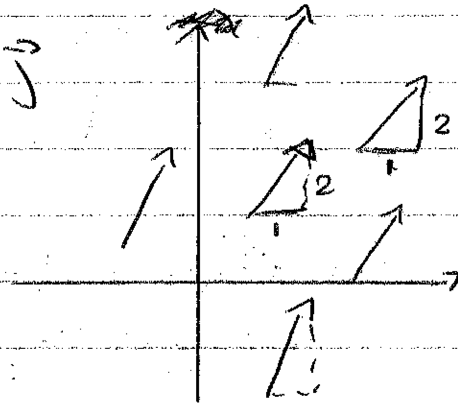
Vector Fields and line integrals.

Any point in space is associated with a vector $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$
or $\vec{F} = \langle M, N \rangle$.

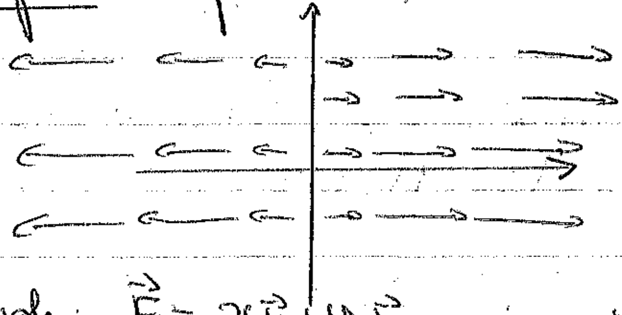
A velocity Field is used to map the wind (magnitude and direction).

A Force Field is used a lot in Physics.

Example: map $\vec{F} = 2\vec{i} + \vec{j}$

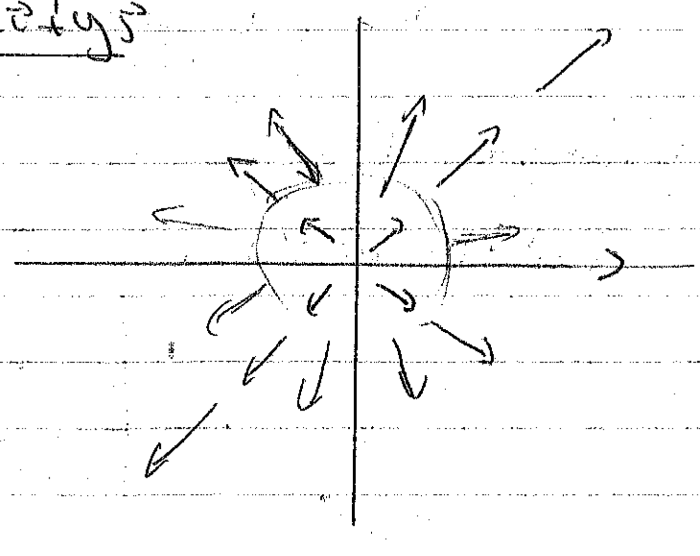


Example map $F = x\vec{i}$

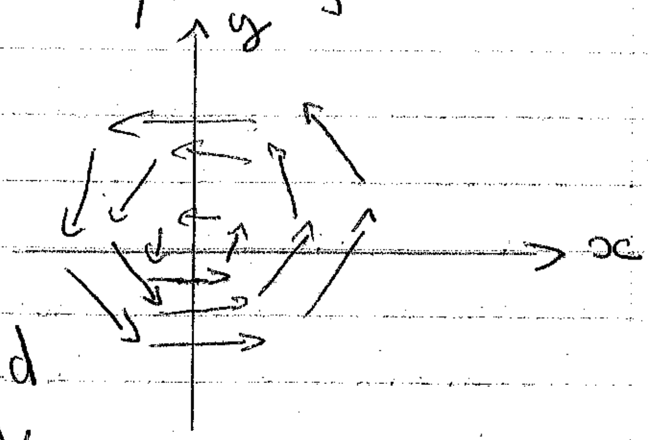


Example: $\vec{F} = x\vec{i} + y\vec{j}$

Magnitude = distance from origin.



<u>Example:</u> $\vec{V} = -y\vec{i} + x\vec{j}$



could represent the velocity of a fluid moving around the origin at a constant speed

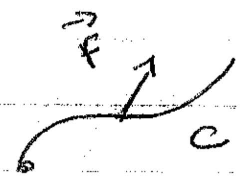
$V = \sqrt{x^2+y^2} = rw$

with $r = \sqrt{x^2+y^2}$

so $w = 1$ angular velocity $= 1$

In Physics the work done by a Force along a trajectory is:

$$W = \int_C \vec{F}.\vec{dr}$$

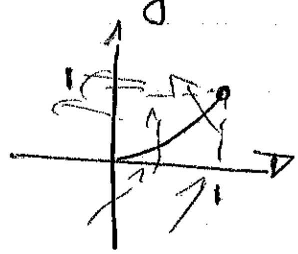


we need to parametrize the curve c with only 1 parameter like t.

$$W = \int_C \vec{F}.\frac{\vec{dr}}{dt}\,dt$$ $\frac{\vec{dr}}{dt}$ is the velocity vector.

<u>Example:</u> $\vec{F} = -y\vec{i} + x\vec{j}$

Find the work done by $\vec{F}$ along C = parabola.



$\begin{cases} x = t \\ y = t^2 \end{cases}$ $\vec{V}\begin{cases} 1 \\ 2t \end{cases}$

$$\vec{F} \cdot \vec{V} = -t^2(1) + t(2t) = -t^2 + 2t^2 = t^2$$

$$\int_0^1 \vec{F} \cdot \vec{V}\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

Another way.

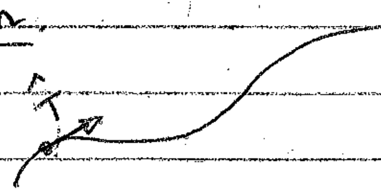
$\vec{F} = \langle M, N \rangle$ $\quad \int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$

$$= \int_C -y\,dx + x\,dy = \int_0^1 x^2\,dx + x(2x)\,dx = \frac{1}{3}$$

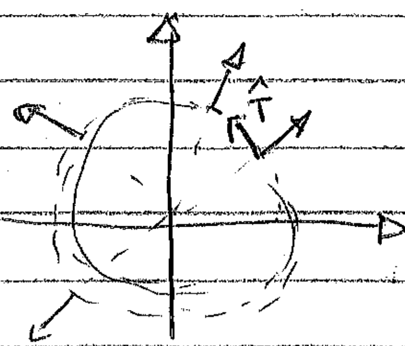
or we can write $\begin{cases} x = t & dx = dt \\ y = t^2 & dy = 2t\,dt \end{cases}$

<u>geometric approach</u>

$$\begin{cases} d\vec{r} = \langle dx, dy \rangle = \hat{T}\,ds \\ \dfrac{d\vec{r}}{dt} = \langle dx, dy \rangle = \hat{T}\,\dfrac{ds}{dt} \end{cases}$$



<u>Example</u>

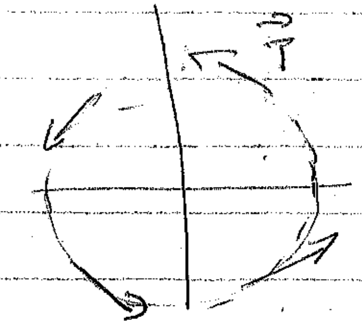


work done around the circle by $\vec{F} = x\vec{i} + y\vec{j}$

$\vec{F} \cdot \vec{T} = 0$

so work done $= 0$

Example: $\vec{F} = -y\vec{i} + x\vec{j}$ } Find work done along the circle.



$\vec{F}\cdot\vec{T} = |\vec{F}| = a =$ radius of circle

$$\text{Work done} = \int_C \vec{F}\cdot\vec{T}\,ds = a\int_C ds = 2\pi a^2$$

$\boxed{\text{Or}}$ $\int_C -y\,dx + x\,dy$ $\quad x = \cos\theta(a)$, $y = \sin\theta(a)$

$$\int -(a\sin\theta)(-a\sin\theta)\,d\theta + a\cos\theta(a\cos\theta)\,d\theta$$

$$= \int a^2(\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi a^2$$

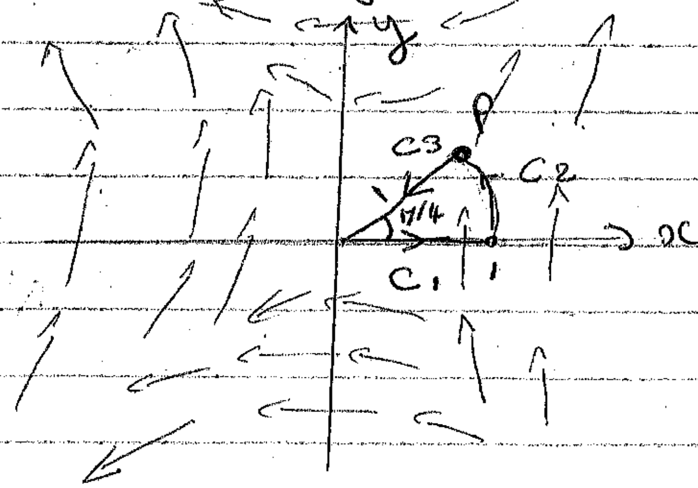
Example: $\vec{F} = y\vec{i} + x\vec{j}$

Find the work done along the path $C_1 - C_2 - C_3$

$P(1/\sqrt{2}, 1/\sqrt{2})$

because $x_P = \cos\pi/4$

$y_P = \sin\pi/4$.



along C_1 $\to$ geometrically $\vec{T} \perp \vec{F}$ so work $= 0$

$\to$ compute the work with $dy = 0, y = 0$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 y\,dx + \int_0^1 x\,dy = 0$$

along C_2

choose θ as a parameter $\begin{cases} x = \cos\theta, & dx = -\sin\theta\,d\theta \\ y = \sin\theta, & dy = \cos\theta\,d\theta \end{cases}$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{\pi/4} \sin\theta\,(-\sin\theta)\,d\theta + \int_0^{\pi/4} \cos\theta\,(\cos\theta)\,d\theta$$

$$= \int_0^{\pi/4} (-\sin^2\theta + \cos^2 d\theta)\theta = \int_0^{\pi/4} \cos 2\theta\,d\theta$$

$$= \frac{1}{2}(+1)\Big[\sin 2\theta\Big]_0^{\pi/4} = \frac{1}{2}$$

along C_3 $\begin{cases} x = t & dx = dt \\ y = t & dy = dt \end{cases}$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{\frac{\sqrt{2}}{2}}^{0} t\,dt \int_{\frac{\sqrt{2}}{2}}^{0} t\,dt = \int_{\frac{\sqrt{2}}{2}}^{0} 2t\,dt = -\frac{1}{2}$$

or : $\begin{cases} x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t \\ y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t \end{cases}$ and integrate

so work done $= 0 + \frac{1}{2} - \frac{1}{2} = 0$