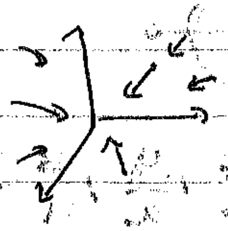
I

Vecta Field in space : $\vec{F} = \langle F_1, F_2, F_3 \rangle$

Examples:

a) force fields like gravitational attraction : like the gravitational attraction of a solid mass at the origin at $(0,0,0)$ on a mass at (x, y, z)



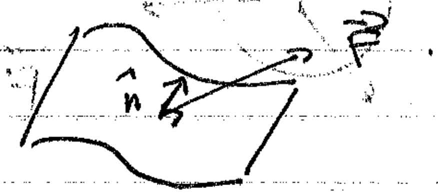
$$\vec{F} = -c\,\frac{\langle x, y, z\rangle}{\rho^3}$$

Electric field and magnetic fields have the same form.

b) Velocity fields $\vec{V}$

c) gradient fieds $\vec{\nabla} u = \langle u_x, u_y, u_z \rangle$

Flux in 3D.



$\hat{n}$ is the unit normal to S. $\hat{n}$ gives the orientation of the surface.

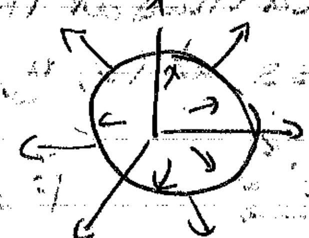
$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n}\, \underline{ds}$$

(surface) area element on S.

$$\vec{ds} = \hat{n}\, ds$$

Example 1

Find the flux of $\vec{F} < x, y, z >$ through the sphere of radius a centered at the origin.



$$\vec{F} \cdot \hat{n} = < x, y, z > \cdot < \frac{x}{a}, \frac{y}{a}, \frac{z}{a} > = \frac{a^2}{a} = a$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = a(4\pi a^2) = 4\pi a^3$$

Example 2 same surface but $\vec{F} = z\vec{k}$

$$\vec{F} \cdot \hat{n} = < 0, 0, z > \cdot < \frac{x}{a}, \frac{y}{a}, \frac{z}{a} >$$

$$\vec{F} \cdot \hat{n} = \frac{z^2}{a}$$

$$\text{Flux} = \iint_S \frac{z^2}{a}\, ds \quad ds = \sin\phi\, a^2\, d\phi\, d\theta$$

$$\text{Flux} = a \int_0^{2\pi}\int_0^{\pi} z^2 \sin\phi\, d\phi\, d\theta \quad z = \cos\phi\, a$$

$$\text{Flux} = a \int_0^{2\pi}\int_0^{\pi} a^2 \cos^2\phi \sin\phi\, d\phi\, d\theta$$

$$\text{Flux} = a^3 \int_0^{2\pi}\int_0^{\pi} \cos^2\phi \sin\phi\, d\phi\, d\theta = \frac{4}{3}\pi a^3$$

How to find $\hat{n}\,ds$ for different surfaces.

1) The surface is a horizontal plane $z = a$

$\hat{n} = \pm \vec{k}$ and $ds = dx\,dy$.

2) Vertical plane $x = a$ $\hat{n} = \pm \vec{i}$

$ds = dx\,dy$

3) sphere of radius a centered at origin.

$$\hat{n} = \pm \frac{\langle x, y, z\rangle}{a}$$

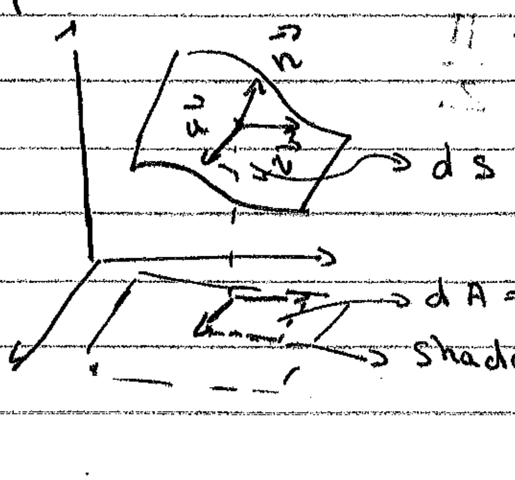
$$dS = \sin\phi\, a^2\, d\phi\, d\theta.$$

4) cylinder of radius a centered on z axis. $\hat{n} = \pm\dfrac{\langle x, y, 0\rangle}{a}$ $ds = a\,dz\,d\theta$

5) if you have the graph of $z = f(x,y)$

then $\hat{n}\,ds = \pm\langle -f_x, -f_y, 1\rangle\, dx\,dy$.

proof



$\vec{u}$: from $\langle x, y, z\rangle$ to $\langle x + \Delta x, y, f(\Delta x + x, y)\rangle$

so $\vec{u} = \langle \Delta x, 0, \frac{\partial f}{\partial x}\Delta x\rangle$

$\vec{V} = \langle 0, \Delta y, \frac{\partial f}{\partial y}\Delta y\rangle$

so $\vec{u} = \langle 1, 0, \frac{\partial f}{\partial x} \rangle \Delta x$

$\vec{V} = \langle 0, 1, \frac{\partial f}{\partial y} \rangle \Delta y$

$$\hat{n}\, ds = \underbrace{\vec{u} \times \vec{V}}_{\substack{\text{area of} \\ \text{parallelogram} \\ \text{normal to } \hat{n}}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \Delta x \Delta y$$

$\hat{n}\, ds = -f_x \vec{i} - f_y \vec{j} + \vec{k}$

$\hat{n}\, ds = \langle -f_x, -f_y, 1 \rangle \Delta x \Delta y$.

Example: Flux $\vec{F} = z\vec{k}$ through the paraboloid $x^2 + y^2 = z$ that lives above the unit disc.

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint \langle 0, 0, z \rangle \cdot \langle -f_x, -f_y, 1 \rangle\, dx\, dy$$

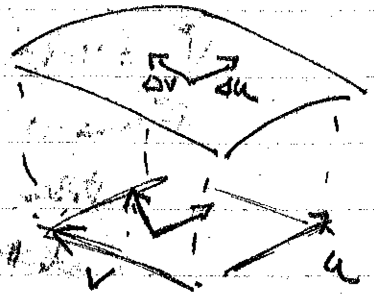
$$= \iint \langle 0, 0, 0 \rangle \cdot \langle -2x, -2y, 1 \rangle\, dx\, dy$$

$$= \iint_D z\, dx\, dy = \iint_D x^2 + y^2\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{\pi}{2}$$

⑥ How to compute $\hat{n}\,ds$ if the surface is parametrize

$$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$



$\vec{r} = \langle x, y, z \rangle = \vec{r}(u,v)$

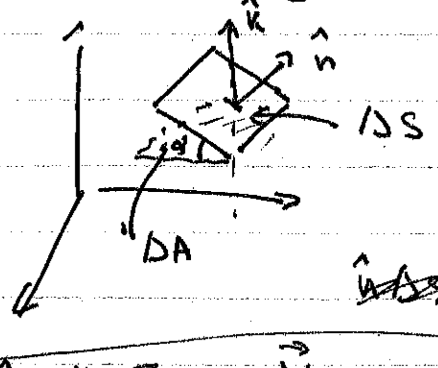
$$\vec{n}\, ds = \left(\frac{\partial \vec{r}}{\partial u} \Delta u \times \frac{\partial \vec{r}}{\partial v} \Delta v \right) = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \Delta u \, \Delta v$$

⑦ How to find $\hat{n}\, ds$ if you don't know the parametrization but we know a normal vector $\vec{N}$ (not necessarily unit) to the surface.

Ⓐ If you have the equation of the plane $ax + by + cd = d$ then $\hat{N} = \langle a, b, c \rangle$

Ⓑ surface is given by the equation $g(x,y,z) = 0$ (level surfaces).

then $\vec{N} = \vec{\nabla} g \perp$ level surfaces. $\Big\}$ given $\vec{n}$ how to compute $\hat{n}\, ds$



$\Delta A = \Delta S \cos\alpha.$

$\cos\alpha = \dfrac{\vec{k} \cdot \vec{N}}{|\vec{N}|\,|\vec{k}|}$ $\qquad \Delta S = \dfrac{\Delta A}{\cos\alpha}$

$\cancel{\hat{n}\Delta S = \dfrac{\Delta A\, \vec{k}\cdot\vec{N}}{|\vec{N}|\,|\vec{k}|}}$

$$\boxed{\hat{n}\, \Delta S = \frac{\pm \vec{N}}{\vec{N} \cdot \vec{k}}\, dx\, dy}$$

with $\hat{n}\Delta S = \dfrac{|\vec{N}|\,\hat{n}\, \Delta A}{\vec{N} \cdot \vec{k}}$

<u>Example</u> S : $\underbrace{z - f(x, y)}_{g(x,y,z)} = 0$

$\vec{N} = \vec{\nabla} g = \langle -f_x, -f_y, 1 \rangle$.

$\frac{\vec{N}}{\vec{N} \cdot \vec{k}} \, dx\,dy = \langle -f_x, -f_y, 1 \rangle \, dx\,dy$