Substitutions changing variables.

2 types of substitution
- → direct New = {old var}
- → inverse old = {New + old.}

I. direct substitution: Bernoulli equation.

$$y' = p(x)\,y + q(x)\,y^n \quad (n \neq 0,\ n \neq 1)$$

divide through by y^n $\Big\}$ $\dfrac{y'}{y^n} = P(x)\dfrac{1}{y^{n-1}} + q(x)$

$V = \dfrac{1}{y^{n-1}}$ ⇒ $V' = +\,y'\dfrac{1}{y^n}(1-n)$ plug into Eqs.

change in variable

$$\boxed{\frac{V'}{(1-n)} = P(x)\,V + q(x)}$$

↳ we get a 1st order linear equation. see previous chapter

Example

$$y' = \frac{y}{x} - y^2 \quad \Big\}\ \text{Bernouilli}$$

divide through by y^2

$$\frac{y'}{y^2} = \frac{y}{x} \times \frac{1}{y^2} - 1$$

$$\frac{y'}{y^2} = \frac{1}{x}\frac{1}{y} - 1$$

$$V = \frac{1}{y} \qquad V' = -\frac{1}{y^2}\,y'$$ play back

$$-V' = \frac{1}{x}V - 1 \quad \text{or} \quad \boxed{V' + \frac{1}{x}V = 1}$$

① integrating Factor:

$$\int p(x)\,dx = \int \frac{dx}{x} = \ln(x) \Rightarrow u = x$$

② multiply through by u

$$(xV)' = x$$

③ integrate

$$xV = \int x\,dx + C$$

$$V = \frac{x}{2} + \frac{C}{x}$$

but $V = \frac{1}{y^{n-1}} \Rightarrow V = \frac{1}{y} \Rightarrow y = \frac{1}{\frac{x}{2} + \frac{C}{x}}$

$$y = \frac{2x}{x^2 + 2C} = \boxed{\frac{2x}{x^2 + C_1}}$$

II inverse substitution = homogeneous 1st order differential equation.

$$y' = F(y/x)$$

Example $y' = \dfrac{x^2 y}{x^3 + y^3} = \dfrac{y/x}{1 + \left(\frac{y}{x}\right)^3}$

or $xy' = \sqrt{x^2+y^2}$ $y' = \sqrt{1 + \left(\frac{y}{x}\right)^2}$

to solve do : $\underline{\underline{y = zx}}$ $(z = y/x)$

$$y' = z'x + z$$

$y' = F(y/x)$ becomes $z'x + z = F(z)$

separate variable to solve.

Example

<u>Example</u>: $y' = \dfrac{y/x + 1}{1 - y/x}$

$z = y/x$ $y = zx$ $y' = z'x + z$

$$z'x + z = \frac{z+1}{1-z}$$

$$z'x = \frac{z + 1 - z + z^2}{1-z} = \frac{1 + z^2}{1-z}$$

$dz\left(\dfrac{1-z}{1+z^2}\right) = \dfrac{1}{x}dx$ (separation of variable)

$$\int \frac{1-z}{1+z^2}\,dz = \int \frac{dz}{1+z^2} - \frac{1}{2}\int \frac{2z}{1+z^2}\,dz$$

$$\Rightarrow \tan^{-1}(z) - \frac{1}{2}\ln(1+z^2) = \ln(x) + C$$

$$\tan^{-1}(z) = \ln\left\{\sqrt{1+z^2}\,x\right\} + C \qquad z = y/x$$

$$\tan^{-1}(y/x) = \ln\sqrt{x^2+y^2} + C$$

In polar coordinates and $x^2+y^2 = r$

$$\tan^{-1}(y/x) = \theta$$

so $\theta = \ln(r+C)$ or $\boxed{r = C_1 e^{\theta}}$